(b) (16 points) Determine the radius and interval of convergence of the power series. You may use your answer from part (a) to assist with checking the endpoints.

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{(k^4+5)^{1/5}}(2x-4)^k.$$

$$f(x) = \sum_{k=1}^{\infty} \frac{(2x-4)^k}{(k^4+5)^{1/5}},$$ find radius of conv (R), IC

→ write the series for $f(x)$ in "standard form", or in powers of $(x-c)$:

$$f(x) = \sum_{k=1}^{\infty} \frac{2^k}{(k^4+5)^{1/5}} (x-2)^k \quad \left(c=2, \ a_k = \frac{2^k}{(k^4+5)^{1/5}}\right)$$

→ apply the ratio test:

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{2^{k+1}}{2^k} \frac{(k^4+5)^{1/5}}{((k+1)^4+5)^{1/5}}$$

1

$$= 2$$

So that $R = \frac{1}{L} = \frac{1}{2}$

→ to find the IC:

$$|x-2| < R \iff -\frac{1}{2} < x-2 < \frac{1}{2}$$

$$\iff \frac{3}{2} < x < \frac{5}{2},$$

So the series definitely converges for all $x \in \left(\frac{3}{2}, \frac{5}{2}\right)$. We need to check convergence at the end points when $x = \frac{3}{2}$, $x = 5/2$.

$$\rightarrow f\left(\tfrac{3}{2}\right)=\sum_{k=1}^{\infty}\frac{2^k\left(-\frac{1}{2}\right)^k}{(k^4+5)^{1/5}}=\underline{\sum_{k=1}^{\infty}\frac{(-1)^k}{(k^4+5)^{1/5}}}$$

apply the AST:

- check for absolute convergence:
  does $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k^4+5)^{1/5}}$ converge or not?

  $\uparrow a_k$

  <u>intuition</u>: looks "almost" like a p-series with $p=4/5\leq 1$, so we expect the series to diverge.

  <u>apply the LCT</u>: compare to $b_k=\dfrac{1}{k^{4/5}}$

$$\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{k^{4/5}}{(k^4+5)^{1/5}}$$

$$=\lim_{k\to\infty}\frac{\cancel{k^{4/5}}^{\,1}}{\cancel{k^{4/5}}\left(1+\cancel{\frac{5}{k^4}}^{\,0}\right)^{1/5}}=1$$

  So both series diverge. (so no absolute convergence)

- Check for conditional convergence:

$$\rightarrow\lim_{k\to\infty}\frac{1}{(k^4+5)^{1/5}}=0\ \checkmark$$

  $\rightarrow$ check that a_k is decreasing

$$a_{k+1}=\frac{1}{((k+1)^4+5)^{1/5}}\leq a_k=\frac{1}{(k^4+5)^{1/5}}\ \checkmark$$

So we get conditional convergence.

(so the series $f(x)$ converges at $x=3/2$)

- Check for convergence at $x = 5/2$

$$f\left(\frac{5}{2}\right) = \sum_{k=1}^{\infty} \frac{2^k \left(\frac{1}{2}\right)^k}{(k^4+5)^{1/5}} = \sum_{k=1}^{\infty} \frac{1}{(k^4+5)^{1/5}}$$

This series diverges by the LCT that we applied above.

- So IC of the series is

$$\left[\frac{3}{2}, \frac{5}{2}\right).$$

Evaluate: $S = \sum_{N=0}^{\infty} \ln\left(\frac{N+5}{N+6}\right)$

(final review sheet #17(c))

→ look at the N^{th} partial sums of the series:

$$S_N = \sum_{k=0}^{N} \ln\left(\frac{k+5}{k+6}\right)$$

$$= \sum_{k=0}^{N} \ln(k+5) - \sum_{k=0}^{N} \ln(k+6)$$

(these sums telescope, or cancel)

$$= \ln(5) - \ln(N+6) \quad (*)$$

→ Now $S = \lim_{N\to\infty} S_N$

$$\overset{\text{by } (*)}{=} \ln(5) - \lim_{N\to\infty} \ln(N+6)$$

$$= -\infty$$

→ So the series S diverges.

Evaluate: $S = \sum_{N=1}^{\infty}\left[\frac{1}{\sqrt{N+3}} - \frac{1}{\sqrt{N+5}}\right]$

(final review sheet #17(e))

→ we expect to get cancellation of most terms in the series

→ let

$$S_1 = \sum_{N=1}^{\infty} \frac{1}{\sqrt{N+3}}$$

$$S_2 = \sum_{N=1}^{\infty} \frac{1}{\sqrt{N+5}}$$

So that $S = S_1 - S_2$

→ $$S_1 = \underset{(N=1)}{\frac{1}{\sqrt{4}}} + \underset{(N=2)}{\frac{1}{\sqrt{5}}} + \sum_{N=3}^{\infty} \frac{1}{\sqrt{N+3}} \quad (*)$$

→ shift the index of the last series in (*):

$M = N-2 \longleftrightarrow N = M+2$

So that

$$S_1 = \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \sum_{M=1}^{\infty} \frac{1}{\sqrt{M+5}}$$

$$= \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + S_2$$

→ Then

$$S = S_1 - S_2 = \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + S_2 - S_2$$

$$= \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} = \frac{1}{2} + \frac{\sqrt{5}}{5}$$

Evaluate: $S = \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}, p>1$

→ let $f(x) = \frac{1}{x(\ln x)^p}$, so that

$$S = \sum_{k=2}^{\infty} f(k)$$

→ we will apply the integral test.
First check the conditions we need to apply the integral test are true:

- $f(k)$ is positive for all $k \geq 2$ ✓
- $f(x)$ is continuous on $[2, \infty)$ ✓
- show $f(k)$ is decreasing for all $k \geq 2$:

$$f'(x) = \frac{d}{dx}\left[f(x)\right] = -\frac{1}{x^2(\ln x)^p} - \frac{p}{x^2(\ln x)^{p+1}}$$

< 0 whenever $x \geq 2$ ✓

→ the integral test tells us that S converges if $I = \int_2^{\infty} f(x)dx$ converges (and diverges otherwise).

→ To see if I converges, apply the u-sub

$$\left(u = \ln(x), du = \frac{dx}{x}; u(2) = \ln(2), \lim_{x\to\infty} u(x) = \lim_{x\to\infty} \ln(x) = +\infty \right)$$

$$I = \int_{\ln(2)}^{\infty} \frac{du}{u^p} = \frac{1}{1-p} \cdot \frac{1}{u^{p-1}}\Bigg|_{\ln(2)}^{\infty}, p>1$$

(<u>Note</u>: $p>1 \longleftrightarrow p-1>0$)

$$= \frac{1}{1-p}\left[\lim_{n\to\infty} \frac{1}{n^{p-1}} - \frac{1}{(\ln 2)^{p-1}}\right]$$

$$= \frac{1}{1-p}\left[0 - \frac{1}{(\ln 2)^{p-1}}\right]$$

$$= \text{constant depending on } p$$

So I converges.

$\rightarrow$ Hence, by the integral test, S converges when $p>1$.

Evaluate: $S = \sum_{k=1}^{\infty} k \cdot \tan\left(\frac{1}{k}\right)$.

→ To determine convergence of S, we apply the n^{th} term test. It says that if $\lim_{k\to\infty} k \cdot \tan\left(\frac{1}{k}\right) \neq 0$, S diverges.

→ evaluate the limit using L'Hopital's rule:

$$L = \lim_{k\to\infty} k \cdot \tan\left(\frac{1}{k}\right)$$

$$= \lim_{k\to\infty} \frac{\tan\left(\frac{1}{k}\right)}{\frac{1}{k}} \quad \left(\frac{0}{0}, \text{ since } \tan(0)=0 \text{ and } \lim_{k\to\infty} \frac{1}{k} = 0\right)$$

$$\overset{\text{by L'Hop.}}{=} \lim_{k\to\infty} \frac{\sec^2\left(\frac{1}{k}\right) \cdot \frac{d}{dk}\left[\frac{1}{k}\right]}{-\frac{1}{k^2}}, \text{ by the chain rule for derivatives}$$

$$= \lim_{k\to\infty} \frac{1}{\cos^2\left(\frac{1}{k}\right)}$$

$$= \frac{1}{\cos^2(0)} = 1 \neq 0$$

→ so by the n^{th} term test, the series S diverges.

Evaluate: $S = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$

→ notice that

$$S = \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}$$

→ we can show that the series converges by comparing with the p-series $b_k = \frac{1}{k^2}$ in the LCT.

→ but we can do even better than that and find the <u>exact value</u> of S by seeing that S is a telescoping series

→ take partial fractions:

$$\frac{1}{(2k-1)(2k+1)} = \frac{A}{(2k-1)} + \frac{B}{(2k+1)}$$

$$\Longleftrightarrow \quad 1 = A(2k+1) + B(2k-1)$$

- when $k = \frac{1}{2}$:

$$1 = 2A \Longleftrightarrow A = \frac{1}{2}$$

- when $k = -\frac{1}{2}$:

$$1 = -2B \Longleftrightarrow B = -\frac{1}{2}$$

So $$S = \frac{1}{2} \sum_{k=1}^{\infty} \left[\frac{1}{2k-1} - \frac{1}{2k+1} \right]$$

→ let $S_1 = \sum_{k=1}^{\infty} \frac{1}{2k-1}$, $S_2 = \sum_{k=1}^{\infty} \frac{1}{2k+1}$, so that

$$S = \frac{1}{2}(S_1 - S_2). \quad (*)$$

→ we expand S_1 as:

$$S_1 = \underset{(k=1)}{1} + \sum_{k=2}^{\infty} \frac{1}{(2k-1)} \quad (**)$$

→ shift the index of the series in (**):

$m = k-1 \longleftrightarrow k = m+1$

So that

$$S_1 = 1 + \sum_{m=1}^{\infty} \frac{1}{2m+1} = 1 + S_2 \quad (***)$$

→ Then (*) and (***):

$$S = \frac{1}{2}\left(1 + S_2 - S_2\right) = \frac{1}{2}.$$

Find a Maclaurin series for the function $g(x) = \frac{x}{(1-x)^3}$.

$\rightarrow$ let $f(x) = \frac{1}{1-x}$.

Notice that $f''(x) = \frac{2}{(1-x)^3}$:

$$\frac{d^{(2)}}{dx^{(2)}}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] = \frac{2}{(1-x)^3}$$

$\rightarrow$ This means that

$$g(x) = \frac{x}{2} \cdot f''(x) \quad (*)$$

$\rightarrow$ we will find a Maclaurin series for $f''(x)$:

$f(x) = \sum_{n=0}^{\infty} x^n$, when $|x| < 1$ by a geometric series expansion with $r = x$.

$$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[x^n\right]$$

$$= \sum_{n=0}^{\infty} n \cdot x^{n-1} = \sum_{n=1}^{\infty} n \cdot x^{n-1} \quad (**)$$

$= \sum_{n=0}^{\infty} (n+1) x^n$, for $|x| < 1$ by shifting the index in $(**)$

$$f''(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (n+1) x^n\right]$$

$$= \sum_{n=0}^{\infty} (n+1) \frac{d}{dx}\left[x^n\right] = \sum_{n=1}^{\infty} n(n+1) x^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) x^n \quad (***)$$

$\longrightarrow$ so by $(*)$ and $(***)$:

$$g(x) = \frac{x}{2} \cdot \sum_{n=0}^{\infty} (n+1)(n+2) x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} (n+1)(n+2) x^{n+1} \quad (****)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} n(n+1) x^n \; , \text{ by shifting the index in } (****)$$

Find the sum of the series

$$S = \frac{\pi}{2} - \frac{\pi^3}{8 \cdot 3!} + \frac{\pi^5}{32 \cdot 5!} + \cdots + \frac{(-1)^N \pi^{2N+1}}{2^{2N+1} (2N+1)!} + \cdots$$

→ Note that for any real t

$$\sin(t) = \sum_{N=0}^{\infty} \frac{(-1)^N t^{2N+1}}{(2N+1)!}$$

→ Taking $t = \frac{\pi}{2}$, we see that

$$S = \sin\left(\frac{\pi}{2}\right) = 1.$$

Use a Maclaurin series to estimate

$I = \int_0^1 e^{-x^2}\,dx$ to within an error of no more than 0.01.

→ first, for any $x \in [0,1]$,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \quad (*)$$

→ Hence, integrating the series in $(*)$ termwise, we find that

$$I = \int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{(2n+1)}\Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} \quad (**)$$

→ Since $(**)$ is an alternating series, for any $n \geq 0$:

$$\left| I - \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(2k+1)} \right| \leq \frac{1}{(n+1)!\,(2n+3)}$$

$$\leq \frac{1}{100}$$

we have imposed this to find large enough n to get the

desired error bound

→ find the smallest $N = 0, 1, 2, \ldots$ so that

$$\frac{1}{(N+1)!\,(2N+3)} \leq \frac{1}{100}$$

- when $N=0$: $\dfrac{1}{1!\cdot 3} = \dfrac{1}{3} \nleq \dfrac{1}{100}$ ✗
- when $N=1$: $\dfrac{1}{2!\cdot 5} = \dfrac{1}{10} \nleq \dfrac{1}{100}$ ✗
- when $N=2$: $\dfrac{1}{3!\cdot 7} = \dfrac{1}{42} \nleq \dfrac{1}{100}$ ✗
- when $N=3$: $\dfrac{1}{4!\cdot 9} = \dfrac{1}{24\cdot 9} \leq \dfrac{1}{100}$ ✓

→ So an approximation to I that is accurate to within $\frac{1}{100}$ of its exact value is:

$$I \simeq \sum_{k=0}^{3} \frac{(-1)^k}{k!\,(2k+1)}.$$